Kaluza-Klein Theory with Torsion

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PASCOS 2009 6-10 July, DESY, Hamburg

- INTRODUCTORY REMARKS
- BASIC FORMALISM
- A SET OF MINIMAL CONSTRAINTS ON TORSION
- CONNECTION COEFFICIENTS
- RICCI TENSOR, RICCI SCALAR
- EINSTEIN EQUTIONS
- SOME RESULTS
- CONCLUDING REMARKS

- WORK DONE IN COLLABORATION WITH KARTHIK H. SHANKAR.

- WE CONSIDER A VARIANT OF 5D K-K THEORY, BUT GENERALIZABLE TO ARBITRARY DIMENSIONS THAT INCLUDES TORSION.

- TORSION, A GEOMETRICAL PROPERTY OF SPACE-TIME, REPRESENTS SPIN DEGREES OF FREEDOM. IN ANALOGY TO EINSTEIN EQUATIONS WHERE THE ENERGY-MOMENTUM OF MATTER FIELDS IS COUPLED TO THE GEOMETRICAL RIEMANNIAN METRIC, WE CAN COUPLE ANGULAR MOMENTUM AND SPIN OF MATTER FIELDS TO THE GEMETRICAL TORSION.

- METRIC AND TORSION TWO INDEPENDENT GEOMETRICAL CHARACTERISTICS OF SPACE-TIME.

- THIS GEOMETRICAL PROPERTY OF SPACE-TIME, *TORSION*, ARISES IN THE DEFINITION OF COVARIANT DERIVATIVE OPERATOR OF VECTOR AND TENSOR FIELDS, FORMING AN ANTI-SYMMETRIC PART OF THE AFFINE CONNECTION COEFFICIENTS (ACCs).

- METRIC COMPATIBILITY LEADS TO THE DETERMINATION OF ACCs IN TERMS OF METRIC COMPONENTS AND THEIR DERIVATIVES, KNOWN AS CHRISTOFFEL SYMBOLS, WHICH ARE SYMMETRIC.

- WITH THE CHOICE OF A SET OF MINIMAL CONDITIONS ON TORSION,WE DETERMINE THE NON-VANISHING COMPONENTS OF TORSION IN TERMS OF METRIC COMPONENTS OF SPACE-TIME

- THE RESULTING ACCs ARE COMPLETELY DETERMINED FROM THE ASSUMED 5D METRIC, LEADING TO SOME REMARKABLE MODIFICATIONS OF THE CONVENTIONAL K-K THEORY

- BASIC FORMALISM; NOTATIONS:

- LET $i, j, k...$ and $A, B, ..$DENOTE COORDINATE AND INERTIAL FRAME INDICES, RESPECTIVELY.

- LET $\mathbf{e}_i = \partial_i$ AND $\theta^i = dx^i$ BE THE BASIS OF THE TANGENT AND DUAL SPACES AT EACH POINT OF SPACE-TIME. LET THE CORRESPONDING INERTIAL BASIS BE $\mathbf{e}_A = e^i_A \mathbf{e}_i$ AND $\theta^A = e^A_i \theta^i$.

- THE VIELBINES e^A_i, e^i_A SATISFY THE ORTHONORMALITY CONDITIONS,

$$e^A_i e^j_A = \delta^j_i, \quad e^A_i e^i_B = \delta^A_B$$

THE METRIC,

$$\mathbf{g}_{ij} = e^A_i e^B_j \eta_{AB}, \quad \mathbf{g}^{ij} = e^i_A e^j_B \eta^{AB},$$

WHERE η_{AB} IS THE MINKOWSKIAN METRIC IN THE INERTIAL COORDINATE SYSTEM.

- COVARIANT DERIVATIVE OPERATOR AND AFFINE CONNECTION COEFFICIENTS:

$$\nabla_{\mathbf{e}_i}\mathbf{e}_j = \tilde{\Gamma}^k_{ij}\mathbf{e}_k, \quad \nabla_{\mathbf{e}_A}\mathbf{e}_B = \omega^C_{AB}\mathbf{e}_C,$$

WHERE $\tilde{\Gamma}^i_{jk}$ AND ω^C_{AB} ARE THE AFFINE AND THE RICCI ROTATION COEFFICIENTS, RESPECTIVELY. FROM THE TRANSFORMATION LAWS BETWEEN THE COORDINATE AND INERTIAL FRAMES, WE HAVE,

$$\omega^A_{BC} = e^i_B(\nabla_{\mathbf{e}_i} e^j_C)e^A_j$$

THE ANTI-SYMMETRIC PART OF THE AFFINE CONNECTION IS THE **TORSION** TENSOR,

$$T^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj}$$

AGAIN, FROM THE TRANSFORMATION LAW BETWEEN THE COORDINATE AND THE INERTIAL FRAMES, WE HAVE THE RELATION,

$$T^i_{jk} = e^B_j e^C_k e^i_A T^A_{BC}$$

- FURTHERMORE, WITH THE STANDARD ASSUMPTION OF METRIC COMPATIBILITY, NAMELY, $\nabla_{\mathbf{e}_i} g_{jk} = 0$, WE OBTAIN,

$$\tilde{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} + K^i_{jk}$$

WHERE

$$\hat{\Gamma}^i_{jk} = (1/2)\mathbf{g}^{im}[\partial_j \mathbf{g}_{km} + \partial_k \mathbf{g}_{jm} - \partial_m \mathbf{g}_{jk}],$$

ARE THE USUAL CHRISTOFFEL COEFFICIENTS (SYMBOLS) THAT ARE SYMMETRIC, AND

$$K^i_{jk} = (1/2)[T^i_{.jk} + T^{.i}_{j.k} + T^{.i}_{k.j}],$$

IS KNOWN AS THE CONTORSION TENSOR.

- 5D SPACE-TIME METRIC:

CONSIDER A FOLIATION OF THE 5D SPACE-TIME IN TERMS OF 4D HYPERSURFACES WITH $x^\mu, (\mu, \nu, ..)$ COORDINATE INDICES ON THE HYPERSURFACES, AND x^5 THE 5D COORDINATE.

$$e^{\mu}_{.a} e^{.b}_{\mu} = \delta^a_b, \quad e^{\mu}_{.a} e^{a}_{.\nu} = \delta^\mu_\nu$$

$$g_{\mu\nu} = e^{.a}_{\mu} e^{.b}_{\nu} \eta_{ab}, \quad g^{\mu\nu} = e^{\mu}_{.a} e^{\nu}_{.b} \eta^{ab},$$

WHERE $e^{\mu}_{.a}$, $e^{a}_{.\mu}$ ARE THE VIERBINES WITH (a, b) AS THE TETRAD INDICES ON THE HYPERSURFACE.

- EXTENSION TO 5D

DEFINE THE 5D VIELBEINES TO BE

$$e^{i}_{A} = (e^{\mu}_{.a}, e^{\mu}_{.\dot{5}}, e^{5}_{.a}, e^{5}_{.\dot{5}}), \quad e^{.A}_{i} = (e^{.a}_{\mu}, e^{.a}_{5}, e^{.\dot{5}}_{\mu}, e^{.\dot{5}}_{5})$$

- EXTENDING THE ORTHONORMALITY RELATIONS, WE ARE LED TO THE UNIQUE CHOICE OF VIELBEINES,

$$e^{\mu}_{\cdot\dot{5}} = 0, \quad e^{5}_{\cdot a} = -e^{\mu}_{\cdot a} A_{\mu} \quad e^{5}_{\dot{5}} = \Phi^{-1}$$

$$e_{5}^{\cdot a} = 0, \quad e^{\cdot\dot{5}} = A_{\mu}\Phi, \quad e_{5}^{\cdot\dot{5}} = \Phi$$

- THE METRIC IN THE 5D SPACE-TIME

$$\mathbf{g}_{ij} = e_{i}^{\cdot A} e_{j}^{\cdot B} \eta_{AB}, \quad \mathbf{g}^{ij} = e^{i}_{\cdot A} e^{j}_{\cdot B} \eta^{AB}$$

$$\mathbf{g}_{\mu\nu} = g_{\mu\nu} + A_{\mu}A_{\nu}\Phi^{2}, \quad \mathbf{g}_{\mu 5} = A_{\mu}\Phi^{2}, \quad \mathbf{g}_{55} = \Phi^{2}$$

$$\mathbf{g}^{\mu\nu} = g^{\mu\nu}, \quad \mathbf{g}^{\mu 5} = -A^{\mu}, \quad \mathbf{g}^{55} = A^{\lambda}A_{\lambda} + \Phi^{-2}$$

- A SET OF CONDITIONS ON THE TORSION:
- CONDITION **1**

$$T^a_{BC} = 0$$

THIS IMPLIES WITH THE GIVEN METRIC, $T^{\mu}_{ij} = 0$;THE ONLY NON-VANISHING COMPONENTS OF TORSION ARE T^5_{ij} .

- CONDITION **2**

$$\omega^{\dot{5}}_{.BC} = \omega^{A}_{.B\dot{5}} = 0,$$

- WHICH IMPLY,

$$\tilde{\Gamma}^{\mu}_{.i5} = 0, \quad \tilde{\Gamma}^{5}_{.i5} = -e^{.\dot{5}}_{5}\partial_i e^{5}_{.\dot{5}}$$

- THE ABOVE TWO CONDITIONS ARE SUFFICIENT TO DETERMINE ALL THE COMPONENTS OF TORSION AND HENCE CONTORSION K^i_{jk}:

$$T^{\mu}_{.ij} = T^{5}_{.55} = 0$$

$$T^{5}_{\mu\nu} = -T^{5}_{\nu\mu} = F_{\mu\nu} + S_{\mu\nu}$$

$$T^{5}_{.\mu 5} = -T^{5}_{.5\mu} = -\Phi^{-2} g_{\mu\sigma}\Gamma^{\sigma}_{.55},$$

WHERE

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad S_{\mu\nu} = A_\mu J_\nu - A_\nu J\mu; \quad J_\mu = \Phi^{-1}\partial\mu\Phi$$

- THE IMPOSED CONDITIONS NOT ONLY DETERMINE THE TORSION IN TERMS OF THE METRIC, BUT MOST IMPORTANTLY, REQUIRE

  $$\partial_5 g_{\mu\nu} = 0$$

  ALL THE HYPERSURFACES IN THE FOLIATING FAMILY HAVE THE SAME 4D METRIC! ALTERNATELY, ONE MIGHT SAY, GRAVITY IS CONFINED TO 4D!!

- NOW, FROM THE GENERAL RELATION,

  $$\tilde{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} + K^i_{jk},$$

  WE FIND, DUE TO REMARKABLE CANCELATIONS,THE RESULTING AFFINE CONNECTION COEFFICIENTS,

  $$\tilde{\Gamma}^\lambda_{55} = \tilde{\Gamma}^\lambda_{\nu 5} = \tilde{\Gamma}^\lambda_{5\nu} = 0,$$

  $$\tilde{\Gamma}^5_{\mu\nu} = \nabla_\mu A_\nu + J_\mu A_\nu,$$

  $$\tilde{\Gamma}^5_{5\mu} = \partial_5 A_\mu + J_5 A_\mu,$$

  $$\tilde{\Gamma}^5_{\mu 5} = J_\mu, \quad \tilde{\Gamma}^5_{55} = J_5, \quad \tilde{\Gamma}^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu},$$

  WHERE $\gamma^\lambda_{\mu\nu}$ IS DERIVED FROM 4D SPACE-TIME WITH METRIC $g^\lambda_{\mu\nu}$; NOTE $\gamma^\lambda_{\mu\nu} \neq \hat{\Gamma}^\lambda_{\mu\nu}$

- SUBSTITUTING THE ABOVE AFFINE CONNECTION COEFFICIENTS IN THE RICCI TENSOR,

$$\widehat{R}_{ik} = \partial_k \tilde{\Gamma}^{j}_{.ji} - \partial_j \tilde{\Gamma}^{j}_{.ki} + \tilde{\Gamma}^{j}_{.km}\tilde{\Gamma}^{m}_{.ji} - \tilde{\Gamma}^{j}_{.jm}\tilde{\Gamma}^{m}_{.ki}$$

  WE FIND,

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}, \quad \tilde{R}_{\mu 5} = \tilde{R}_{5\mu} = \tilde{R}_{55} = 0,$$

  WHERE $R_{\mu\nu}$ REPRESENTS THE RICCI TENSOR ON THE TORSION-FREE 4D SPACE-TIME. NOTE ALSO RICCI SCALAR IN 5D $\tilde{R}= R$, THE 4D TORSION-FREE RICCI SCALAR.

- REMARK:THE RESULTS ARE NOT SPECIFIC TO 4 AND 5 DIMENSIONS. IT IS STRAIGHT FORWARD TO EXTEND THEM TO ARBITRARY D AND D+1 DIMENSIONS.

- DISCUSSION OF SOME RESULTS:

- GEODESIC EQUATIONS

$$\ddot{x}^5 + \tilde{\Gamma}^5_{\cdot\,\mu\nu}\dot{x}^\mu\dot{x}^\nu + \left(\tilde{\Gamma}^5_{\cdot\,\mu 5} + \tilde{\Gamma}^5_{\cdot\,5\mu}\right)\dot{x}^\mu\dot{x}^5 + \tilde{\Gamma}^5_{\cdot\,55}\left(\dot{x}^5\right)^2 = 0,$$

$$\ddot{x}^\lambda + \Gamma^\lambda_{\cdot\,\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

THE GEODESIC EQUATIONS ALONG HYPERSURFACE ARE EXACTLY THE SAME AS THE GEODESIC EQUATIONS IN THE TORSION FREE 4D SPACE-TIME, IN CONTRAST WITH THE CONVENTIONAL KALUZA-KLEIN THEORY

- EINSTEIN'S EQUATIONS; THE 5D EQUATIONS SPLIT INTO,

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R - (1/2)A_\mu A_\nu \Phi^2 R = \mathbf{T}_{\mu\nu}$$

$$-(1/2)A\mu\Phi^2 R = \mathbf{T}_{\mu 5}, \quad -(1/2)\Phi^2 R = \mathbf{T}_{55}$$

- VACUUM SOLUTIONS IN 5D SPACE-TIME,

$$\mathbf{T}_{ij} = 0,$$

IMPLIES $R = 0$ AND $R_{\mu\nu} = 0$. HENCE, 4D METRIC COMPONENTS, $g_{\mu\nu}$, OBTAINED FROM VACUUM SOLUTIONS IN 5D SPACE-TIME ARE IDENTICAL TO THOSE FROM THE VACUUM SOLUTIONS IN THE TORSION FREE 4D SPACE-TIME, AND CONVERSELY, ALL POSSIBLE 4D

VACUUM SOLUTIONS ARE VACUUM SOLUTIONS IN THE 5D SPACE-TIME.

- IN THE CASE OF NON-VACUUM SOLUTIONS, THE SITUATION IS DIFFERENT. DISCUSS

- CONSIDER, FOR INSTANCE, THE PSEUDO-VACUUM SOLUTIONS, WHERE ONLY THE 4D STRESS-TENSOR $\mathbf{T}_{\mu\nu} = 0$. TAKING THE TRACE OF THE FIRST EINSTEIN EQUATION,

$$R(1 + (1/2)A_\mu A_\nu \Phi^2) = 0,$$

WHICH IMPLIES, FOR NON-VANISHING R, THE SOLUTION REQUIRES $A\mu$ TO BE A TIME-LIKE VECTOR.

- THUS 5D PSEUDO-VACUUM SOLUTIONS COULD YIELD NEW SOLUTION IN TORSION FREE 4D SPACE-TIME. POSSIBILITY OF A SPHERICALLY SYMMETRIC PSEUDO-VACUUM SOLUTION DIFFERENT FROM THE USUAL SCHWARZSCHILD SOLUTION.

- MODIFIED FRIEDMAN-ROBERTSON-WALKER (FRW) EQUATIONS

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\left[\frac{2\rho - 3\mathrm{P}A_0^2}{2 - A_0^2}\right], \quad \frac{\ddot{a}}{a} = -\frac{8\pi}{3}\left[\frac{\rho + 3\mathrm{P}(1 - A_0^2)}{2 - A_0^2}\right]$$

- WITH $A_0 = 0$, WE RECOVER THE USUAL EQUATIONS OF THE FRW STANDARD MODEL WITH ACCELERATION ALWAYS NEGATIVE.

- WITHE THE RELATIVISTIC EQUATION OF STATE $P = \rho/3$, THE MODIFIED FRW EQUATIONS REMAIN INDEPENDENT OF A_0 AND REMAIN IDENTICAL TO THE ORIGINAL FRW EQUATIONS.

- HOWEVER, WHEN P IS NOT EQUAL TO $\rho/3$, THE MODIFIED EQUATIONS GIVE RISE TO THE INTRIGUING POSSIBILITY OF THE EMERGENCE OF POSITIVE ACCELERATION !

- CONCLUDING REMARKS:

- TORSION IS AN INTEGRAL, GEOMETRICAL PROPERTY OF SPACE-TIME,

- THE MODEL WE HAVE EXPLORED HAS A VERY GENERAL MATHEMATICAL RESULT PERTAINING TO THE USUAL METRIC AFFINE CONNECTION COEFFICIENTS AND TORSION,

- IN HIGHER DIMENSIONAL THEORIES, IT PROVIDES AN ALTERNATIVE WAY TO CONFINE GRAVITY, YET MODIFYING IT IN A SIGNIFICANT WAY TO BE RELEVANT TO ASTROPHYSICS AND COSMOLOGY,

- WE HAVE EXPLORED QUALITATIVELY SOME OF THE CONSEQUENCES. NEEDS MUCH FURTHER WORK.